



Ising Model at the Phase Transition Point

Clément Hongler

Ecole Polytechnique Fédérale de Lausanne

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CLÉMENT HONGLER

Chair of Statistical Field Theory, Institute of Mathematics, Ecole Polytechnique Fédérale de Lausanne

1. INTRODUCTION

The developments of statistical mechanics and quantum field theory are among the major achievements of the 20th-century science. In the second half of the century, these two subjects started to converge, notably thanks to ideas of the renormalization group (see e.g. [Car96, Mus09]). In two dimensions, the result of this convergence is one of the most remarkable chapters of mathematical physics (see e.g. [Mus09]): the two-dimensional conformally symmetric field theories show remarkable structures that enables one to understand them at level of resolutions that are completely unprecedented, making such theories powerful building blocks of many contemporary subjects of mathematics and physics.

Unfortunately, while being at the heart of much of current research in physics and related fields, this picture is not mathematically rigorous: a framework to understand renormalization group ideas does not exist at the moment, most of the relevant quantum field theories have not yet been constructed in a mathematically satisfying way, and there are few clear connections between non-trivial statistical mechanics models and quantum field theories. Identifying the relevant objects to understand such connections and being able to phrase them in mathematical language, beyond mere physical intuition, is hence a fundamental research problem.

At the beginning of this century, several mathematical breakthroughs shed new light on this picture: the introduction of SLE curves (see e.g. [Sch07]), and of lattice analysis techniques (see e.g. [Smi10b]) gave hope for a rigorous approach to planar lattice models and for the construction of mathematically precise bridges between such models and well defined continuous objects.

My current work is mainly dedicated on making this connection precise. In particular, my approach is centered around the Ising model, and the field theories describing its limit. Nine years ago, this connection was largely conjectural, and many aspects of it were considered as ‘black magic’ by most mathematicians. Now, we are getting closer to a complete understanding of this picture. This is to date one of the very few natural examples of a rigorous bridge between a lattice model and a conformal field theory.

1.1. Ising Model. The Ising model [Isi25] is the most fundamental model of equilibrium statistical mechanics, studied by over 12000 papers in the last decades. It is a model that is extremely simple and concrete, making it suitable to applications in various areas of science. Progress on this model has triggered the development of many areas of applied sciences, theoretical physics and pure mathematics.

The Ising model consists of random ± 1 -spins σ living on a graph G . The probability of a configuration of spins $\sigma \in \{\pm 1\}^G$ is proportional to $e^{-\beta H(\sigma)}$, where the energy $H(\sigma) = -\sum_{i \sim j} \sigma_i \sigma_j$ (the sum is over all pairs of adjacent vertices) and $\beta > 0$ is the inverse temperature.

The central question in statistical mechanics is to describe the large-scale behavior of the system: if we take G to be Ω_δ , a discretization of a planar domain Ω by a grid of mesh size δ , what happens as $\delta \rightarrow 0$? As was shown by Onsager [Ons44], a phase transition occurs at the critical parameter

value $\beta_c = \frac{1}{2} \ln(\sqrt{2} - 1)$: for $\beta < \beta_c$, the system is disordered (no long-range alignment effect), while for $\beta > \beta_c$ a long-range order takes place (there is a global alignment phenomenon).

At $\beta = \beta_c$, the model is expected to converge to a universal, conformally invariant scaling limit as $\delta \rightarrow 0$ [BPZ84b, Car84]. This phase is the most interesting one from both mathematical and physical viewpoints and has been the heart of the matter for the last fifty years. New ideas of discrete analysis have come in the last decade and given the hope to understand this phase from a field-theoretic (rather than thermodynamic) viewpoint, taking into account the emergent symmetries of the model.

1.2. Main Questions. The questions I have worked on for the past years can be briefly summarized as:

- What are scaling limit of the fields and the curves of the Ising model in two dimensions?
- What are the algebraic, analytic and geometric structures which describe them?

These questions came together with a number of spectacular conjectures from the physics literature; now, most of these conjectures have been proved and we have found some surprises.

In the recent years, my research has been refined into the following ideas:

- How can we make sense of the very ideas of conformal field theories? Previously, we proved the predictions coming from physical techniques, but the proof turned out to be fairly different from the original physical derivation. The picture suggested by conformal field theory is very general and appealing, as it contains beautiful algebraic and geometric ideas; understanding it would moreover shed some light on quantum field theory as a whole.
- How can we enlarge the picture? How can we establish the convergence of the whole set of the fields and curves, not just distinguished ones? How can we understand how these fields are organized together? Where do the new ideas introduced to solve the problems lead us?

This plan has already shown a number of remarkable successes and it has put us in position to attack many fascinating challenges.

Ultimately, the study of the Ising model has significantly improved the overall mathematical understanding of the connection between statistical mechanics and quantum field theories: it is the first time that the convergence of a natural lattice model to a minimal model of conformal field theory has been established.

2. MAIN RESULTS

Much of my research has consisted in developing new techniques to prove conjectures coming from the mathematical and physical literature about the 2D Ising model scaling limit. In particular, spectacular formulae were predicted, as well as conformal symmetry for a variety of objects. I discuss here the main achievements in this direction.

2.1. Spin Correlations. Spin correlations, which describe how the Ising spins influence each other across the lattice, have been the main question in the subject for the last sixty years and have direct experimental relevance [Mus09]. With Izyurov and Chelkak [CHI15], we solved an important problem that was open for 45 years: the spin correlations with bounded geometries. We showed that the critical Ising spin correlations have a conformally invariant scaling limit and explicitly computed all their n -point functions: we obtained the conformal invariance of all the correlations of the spin field $\delta^{-1/8}\sigma$ (where δ is the mesh size and σ is the ± 1 -valued spin field on Ω_δ , see Section 1.1). Our results include the proof of a number of celebrated formulae appearing in Conformal Field Theory, and culminating in the conjectures of Cardy [Car84] and of Burkhardt and Guim [BuGu93]. As a particular case, we obtain the following result (conjectured by Cardy):

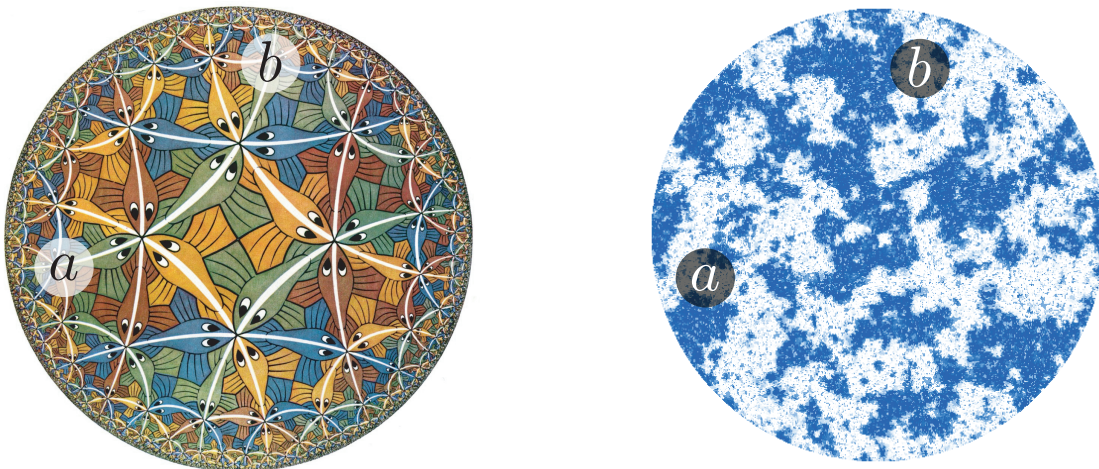


FIGURE 2.1. Correlations of Ising spins (right) at a, b can be expressed in terms of the hyperbolic metric, depicted by this picture of M.C. Escher (left): η_a, η_b become (informally) proportional to the size of the fish at a, b and $d(a, b)$ is (informally) related to the number of fish to go from a to b .

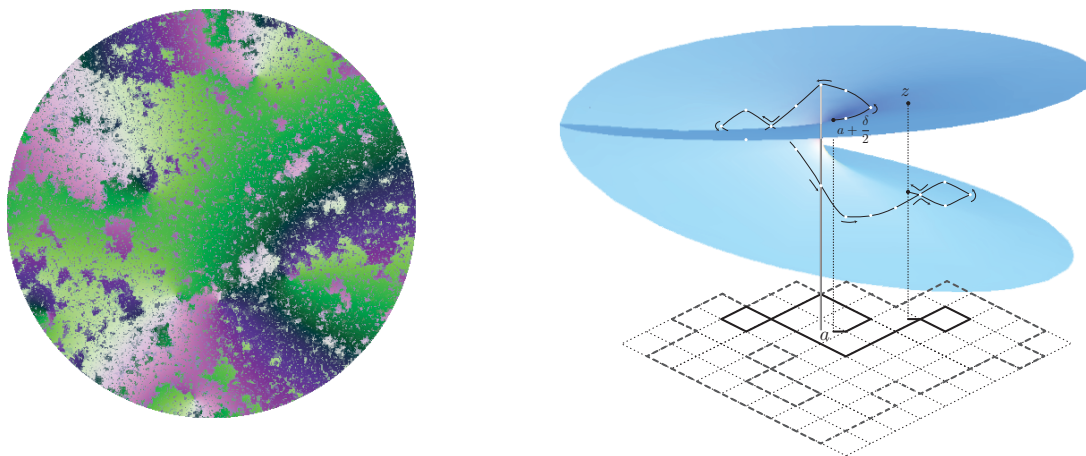


FIGURE 2.2. Ising spinors, which allow to compute spin correlations, can be defined in terms of a twist of the model (left), as complex sums calculated on a double cover of the lattice (right).

Theorem 1 ([CHI15]). *Consider the critical Ising model on a square lattice discretization Ω_δ of a simply-connected domain Ω . For any $a, b \in \Omega$, we have, as $\delta \rightarrow 0$,*

$$\frac{1}{\delta^{\frac{1}{4}}} \mathbb{E}_{\Omega_\delta} [\sigma_a \sigma_b] \xrightarrow{\delta \rightarrow 0} \frac{C \eta_a^{\frac{1}{8}} \eta_b^{\frac{1}{8}}}{(e^{2d(a,b)} - 1)^{\frac{1}{4}}}$$

where η_a, η_b denote the length elements of the hyperbolic metric at Ω at a, b (i.e. the inverse conformal radii), and where d_Ω denotes the hyperbolic distance between a and b , and C is an explicit constant.

The heart of our proof relies on a delicate connection between the spin correlations and a discrete holomorphic spinor (see also [Hon13]). Here we introduce a new spinor observable, which allows one to rephrase the exact solvability of the model in terms of discrete holomorphicity, and the quantities of interests in terms of boundary value problems. Our paper translates the quantities of interest in terms of special values of a discrete holomorphic spinor. The results are then obtained by establishing the convergence of this spinor, a delicate analysis of their behavior near their singularities: in particular, this involves a new use of the symmetries of the model, together with the construction of new kernels for functions with monodromies, and the use of Beurling estimates for random walks [LaLi04]. This enables one to study the logarithmic derivatives of the spin correlations. A new technique based on decorrelation inequalities and discrete complex analysis is then used to calibrate the correlations.

This paper, which is one of the first proofs of the convergence of a lattice model field to a non-Gaussian conformally symmetric random field. It is the key building block for a number of constructions, in particular the probabilistic construction of the continuous critical Ising magnetization field [CGN15a, CGN16], and more generally the study of the local fields of the Ising model [GHP16, HKV17].

From the point of view of connecting the Ising model with Conformal Field Theory, this is a landmark result, as any correlation function can in principle be expressed through the fusion of several spins. The spin correlations being the most natural quantity associated with the Ising model, this also paves the way for many new results in applied sciences. The results in this paper have had a number of applications [CGN15a, CGN16, CGN14, CSZ16]. The methods have also paved the way to further developments (see Sections 2.10 and 2.11).

2.2. Boundary Correlations. In my Ph.D. thesis [Hon10], I obtained conformal invariance results for boundary spin correlations, showing a very different behavior from the bulk behavior, as had been predicted by physicists. With Kytölä [HoKy13], we developed a toolbox to study rather general boundary correlations, in particular those living on rough geometries: this involves in particular the representation of discrete holomorphic functions in terms of convolution kernels, which then allows one to study the ratios of such functions on arbitrary boundaries.

In addition to its intrinsic interest, this result is instrumental in the proof of new convergence results for Ising interfaces (see Section 2.5) and has promising applications to the so-called stress-energy tensor of the model. The structure of the boundary correlation has furthermore enabled the proof that multiple Ising interfaces converge to multiple SLE(3) [Izy17] and has seen interesting applications to the study of random currents [Lis17].

2.3. Energy Correlations. With Smirnov [HoSm13], we studied the energy correlations, one of the main questions in the subject: they describe how the energy $H = -\sum_{i\sim j} \sigma_i \sigma_j$ is distributed as a function of the geometry of the domain. We obtained a simple formula, relating the average energy density to conformal geometry: in simply-connected domains, the energy density is proportional to the hyperbolic length element. This proved predictions from boundary conformal field theory of the 1980s. I generalized this result in my Ph.D. thesis [Hon10] to study arbitrary correlations on multiply-connected domains, obtaining a Pfaffian structure.

This result relies on the study of fermionic correlators: these correlators allow one to formulate the quantities of interests at their singularities, and to translate the energy correlations in terms of discrete boundary value problems. The convergence uses a number of discrete complex analysis techniques [CGS16, Ken00, Ken01, Smi10a], ideas of Kadanoff and Ceva about local operators [KaCe71], and a new formulation of such problems, together with new inequalities.

In addition to pioneering a new side of the investigation of the conformal invariance of the Ising model (the local field picture), this work led to interesting consequences. First, it suggested that, somehow surprisingly, the renormalized energy field, despite having non-trivial finite correlations, cannot be given a usual probabilistic sense as a random Schwartz distribution: the covariance kernel

is too singular for this. The results have moreover found a number of applications and generalizations, in particular the work of Giuliani, Greenblatt and Mastropietro [GGM12] on non-integrable versions of the Ising model. Finally, the investigation of the relevant boundary value problems that naturally arise in the solution of this problem has led to new complex analysis ideas and methods (see Section 2.9).

2.4. Chordal Interfaces and SLE. The breakthrough that initiated the rigorous study of conformal invariance was the introduction of Schramm’s SLE curves in 1999. The motivation was to study random curves arising in lattice models, the Ising model being one of the prime examples, conjectured to converge to SLE in the scaling limit (i.e. as $\delta \rightarrow 0$, see Section 1.1). For instance, the interfaces between $+$ and $-$ spins generated by boundary conditions should converge to SLE (3). The general vision to study random curves arising in lattice models, consists in combining crossing inequalities with conformal martingales; this vision was realized in our paper with Chelkak, Duminil-Copin, Kemppainen and Smirnov [CDHKS14], where we prove the convergence of Ising interfaces to SLE (3) and the convergence of the FK-Ising interfaces to SLE (16/3).

The two important ingredients of the proofs are the precompactness technology of the proof are RSW estimates proven by Duminil-Copin, Nolin and myself (see Section 2.6) and the scaling limit of fermionic observables shown by Chelkak and Smirnov [ChSm09].

This result has led to many applications (e.g. [Izy17, KeSm15, KeSm16, Wu16a, Wu16b, Wu16c]), and the convergence to SLE (16/3) is also instrumental in the proof of the free boundary conditions case as well (see Section 2.5).

2.5. Free Boundary Conditions and SLE. This result on chordal interfaces the Ising model was generalized by Kytölä and myself to include free boundary conditions, thus proving a conjecture of Bauer, Bernard and Houdayer [BBH05].

This paper introduced a number of innovative techniques: a construction of martingales to control the growth of the interfaces, the connection of these martingales to correlation ratios, using the Kramers–Wannier duality, a path-integral decomposition of the correlations in terms of simpler ones, their averaging using SLE techniques and physics-based formulae, and the development of a discrete complex analysis framework to handle boundary correlations on rough boundaries.

Our result has allowed us to prove the conformal invariance of crossing probabilities with free boundary conditions, the construction of a ‘Free Arc Ensemble’ (see Section 2.7). In turn this has been decisive in understanding the ‘full scaling limit’ convergence to the CLE processes (see Section 2.8 below). Our result has also seen a number of interesting applications [Wu16a, Wu16b, QiWe17], and a new alternative proof has been proposed [Izy15]. Furthermore, the discrete complex analysis technology is relevant to the study of the Ising boundary stress tensor (in progress).

2.6. Crossing Estimates. A central tool in stochastic geometry, which allows one to transfer results between the discrete and the continuum, is crossing probability estimates. Such bounds are crucial in percolation, for instance, where they are at the root of the Russo–Seymour–Welsh theory. With Duminil-Copin and Nolin [DHN11] we obtained such bounds for the random-cluster representation of the Ising model. Our bounds have proven to be very useful for a number of applications, including the convergence to random curves. They lie at the heart of the proof by Lubetzky and Sly [LuSl10] about the mixing times of an Ising model Markov chain, which solves a long-standing conjecture. The proof relies on discrete complex analysis, second moment methods and random walk estimates.

With Chelkak and Duminil-Copin [CDH16], we were able to generalize this result to arbitrarily rough domains, getting interesting connections with electrical resistance, and opening the way to complete the program of a full understanding of Ising curves (see Sections 2.7 and 2.8). The proof relies on a delicate combination of ideas from [DHN11], together with factorization estimates

coming from discrete geometric function theory [Che16] and on modification of multipoint FK-Ising observables.

2.7. Spontaneously Generated Interfaces. The interfaces that are proven to converge to SLE are usually generated by specific boundary conditions, which force the location of their start- and end-points. It is natural to ask about spontaneously generated interfaces.

Building on the work with Kytölä on free boundary conditions (see Section 2.5) and on the work with Chelkak and H. Duminil-Copin about ‘rough’ crossing probabilities (see Section 2.6), with Benoist and Duminil-Copin [BDH16], we have identified such curves. The proof relies on an identification of the excursions of the process, together with a careful estimation of the dimensions of the zeroes of Bessel processes, enabling one to control the behavior of the process outside its excursions. As a corollary of our result, we were able to prove the conjecture of Langlands, Lewis and Saint-Aubin [LLS00] about conformal invariance and universality of crossing probabilities. This has seen interesting applications (see eg. [Wu16b, Wu16c]), and most importantly, this result is instrumental in the proof of convergence of the Ising interfaces to CLE(3) (see Section 2.8).

2.8. Conformal Loop Ensembles. Conformal Loop Ensemble (CLE) processes were introduced by Sheffield [She09] as the natural candidates for the conformally invariant scaling limit the entire sets of macroscopic curves arising in a lattice model. Thanks to a number of results pertaining to their characterization and construction [ShWe12], these loop ensembles have become increasingly important in the last years. One of the prime examples of this correspondence is the convergence of the Ising interfaces to CLE(3), which was for a long time conjectural.

With Benoist, we have recently proved this conjecture, and established that the loop ensembles with + boundary conditions converge to the CLE(3) processes [BeHo17]. The idea is to use the coupling between the random cluster model and the Ising model, in order to make a ‘twist’ of boundary conditions, that allows one to access these loops. A number of precise estimates are needed to make sure that there is no ‘dust’ (i.e., collections of microscopic interfaces with a macroscopic effect) affecting the process, which relies on rough boundary crossing estimates (see Section 2.6).

This result is a cornerstone for the investigation of the complete sets of macroscopic interfaces with arbitrary boundary conditions (in progress). In addition to its intrinsic interest, our result leads to a number of very nontrivial consequences, in particular thanks to the loop-soup representations [LaWe04, ShWe12] of CLE processes (see e.g. [QiWe17]).

2.9. Complex Analytic Data. The results obtained here all rely to some extent on complex analysis. With Phong, we studied the continuous complex-analytic problems involved and obtained some interesting results about Hardy spaces. The key result, inspired by Ising model ideas, is the invertibility of a certain operator of Steklov–Poincaré type [HoPh13]. The proof relies on a tree-like decomposition of this operator into other operators, and by revealing a quadratic form that makes them positive definite.

With Kytölä and Zahabi [HKZ14], we studied similar problems from a discrete perspective. We obtained various results, connecting the Ising transfer matrix to a discrete analytic continuation operator, thus explaining the success of the method. We also explained how discrete Steklov–Poincaré operators can be composed to glue domains and to encode geometric data in a holographic way. This complex-analytic formulation of the Ising algebraic structures led to a clarification of the rôle of discrete holomorphicity in the study of the model; furthermore, it has initiated the investigation of number of Conformal Field Theory structures at the lattice level from discrete holomorphicity (see Section 2.11 below).

2.10. Lattice Local Fields. Thanks to the above results, we now know that the spin field and the energy field (the product of two adjacent spins) have scaling limits, corresponding to two local fields of conformal field theory. What about other functions of (a finite number of) spins, and other local fields of conformal field theory? The investigations performed so far suggest that one can understand

all the local fields of conformal field theory as functions of finitely-many spins, which we call ‘lattice local fields’.

With Gheissari and Park [GHP16], we have done a first step in this direction: we have obtained the first-order terms in the asymptotic behavior of lattice local fields. The toolbox of this article, together with the Virasoro structures found at the lattice level (see Section 2.11) and upcoming results involving spin-energy-fermion correlations, can be expected to ultimately lead to the complete correspondence between lattice local fields and Conformal Field Theory local fields.

2.11. Conformal Field Theory at the Lattice Level. The space of local fields of Conformal Field Theory (CFT), which is conjecturally the lattice local fields scaling limits (see Section 2.10) possesses a very interesting algebraic structure. This space can be split into three irreducible representations of the Virasoro algebra [BPZ84b]. In a sense, the Virasoro algebra is the algebraic translation of conformal symmetry. A natural question is how much of this structure can be found at the lattice level, on the space of lattice local fields.

Surprisingly, with Kytölä and Viklund [HKV17], we found that we can reveal the full Virasoro symmetry at the lattice level, as a consequence of the exact solvability. This relies on a number of new ideas, in particular on discretizing a string theory construction of the Virasoro algebra, known as the Sugawara construction: using a number of tools [CHI15, GHP16] and developing new ones, we construct lattice analogues of the contour integrals, fermions and monomials used in the Sugawara construction, and show that the exact commutation relations that exist in the continuum are already present at the lattice level.

This answers a question that was asked by Itoyama and Thacker [ItTh87], about the connection between discrete holomorphicity and elaborated algebraic structures, in particular those of the theory of Vertex Operator Algebras [Fuc95]. Furthermore, this work allows us to complete the correspondence between lattice local fields and CFT local fields. Thanks to the elementary and concrete nature of the Virasoro action in this framework, this result represents important progress in the programme to rigorously connect statistical mechanics with Conformal Field Theory. On one hand, it allows one to restore the probabilistic view that is lost in most of axiomatic field theory, together with the symmetries which were only expected in the continuum. On the other hand, it shows that many of the structures of the theory of Vertex Operator Algebras can be seen as arising from simple principles on the lattice level.

2.12. Perspectives. Previous and current work allows the formulation of numerous further challenges, including revolutionary ideas at the intersection of statistical mechanics and field theory, which now can be expected to find new rigorous statistical mechanics applications [KaCe71, SMJ77, Car84, FQS85b, Zam89]. A long-term goal is to give precise, well-defined, lattice incarnations to such ideas, to breathe probabilistic life into them. This is necessary for such ideas to find their way to concrete problems, and it will also enrich probability with fascinating new structures.

2.12.1. Minimal Models. It is a major challenge to extend the results that have been obtained for the Ising model to other models and field theories, which seems to require significant new ideas. An important question that is very appealing is to try to construct the field theories (which have no probabilistic existence for the moment) into questions about SLE (which are always well defined). On one hand, the SLEs form a one-parameter family, while the minimal model field theories form a discrete series, corresponding to a discrete subfamily of the SLE processes. What is special about them?

We are working on finding what is special about certain SLEs that makes them relevant in constructing minimal models. In particular, we gathered evidence suggesting that $SLE_{16/5}$ has a special structure, that allows one to link it with the first supersymmetric minimal model. A promising idea is the relevance of SLE to construct macroscopic disorders, which should allow one to construct fermions.

2.12.2. *Vertex Operator Algebras.* The construction used to construct the Virasoro action for the Ising model can be generalized to other models. In particular, the discrete Gaussian free field also carries a Virasoro action on the lattice level. What other models carry such an action?

Preliminary results seem to indicate that theories with affine Kac–Moody symmetries would be amenable to such constructions, through the so-called Wakimoto construction. The possibility of understanding such theories on the lattice level would help bring a concrete sense to such models, and ultimately allow us to understand them better as statistical mechanics models, rather than abstract algebraic or geometric objects.

2.12.3. *Thermal Perturbations and Isomonodromy.* What happens away from the critical point for the Ising model? Meaningful scaling limits can be obtained if one lets the temperature get closer to the critical value as one rescales the system at an appropriate speed. In this case, one obtains a massive field theory and conformal symmetry is broken. Remarkably, some exact computations are still possible [WMTB76]. Ideas of Sato, Miwa and Jimbo [SMJ77, PaTr83, Pal07] have suggested how to approach the emergent integrability as an isomonodromy problem. The Painlevé function describing spin correlations arises most elegantly as a compatibility property for some bundle of solutions to a certain family of boundary value problems.

We are working on realizing the isomonodromy approach within the framework of discrete analysis. Our approach will allow us to understand two issues. First, the effects of boundaries on isomonodromy: previously there were no boundary conditions, but with the technology we developed, they become accessible. Second, we want to understand the correlations of the Virasoro descendent: thanks to our lattice constructions, those are simply correlations of lattice local fields, and the question of the scaling limit becomes natural. This would allow one to make concrete sense of and rigorously prove some beautiful intuitions of Cardy and Mussardo [CaMu90] for the descendent correlations.

2.12.4. *Magnetic Perturbations, Scattering Matrices and E_8 Algebra.* The Ising model has two relevant perturbations: one is thermal (discussed above) and the other consists in breaking the ± 1 symmetry by adding a magnetic field, in which case it is widely believed that the model loses its lattice integrability. Still, massive limits exist, as was recently shown rigorously by Camia, Garban and Newman [CGN16], based on our work about SLE and spin fields.

One of the most mysterious and intriguing ideas of the late 1980s was the proposal of Zamolodchikov [Zam89] of a way of describing the massive limit. He suggested looking at scattering matrices of the massive Lorentzian field theories and suggested that the scattering matrix corresponding to the Ising model with a magnetic field was purely elastic, contained eight particles and was related to the E_8 Lie algebra. This has raised a lot of attention, including some experimental successes into checking some consequences of this conjecture (Coldea et al. [Col10]). This was further developed by many physicists, in particular Delfino and Mussardo [DeMu95], who produced an impressive range of exact predictions based on this idea.

For a long time, it seemed unreasonable to attack this conjecture mathematically and to ‘find E_8 ’ within the Ising model. However, we recently made some significant progress in this direction and it now appears perfectly realistic to approach it. If successful, this project will be the first time the exceptional algebra E_8 is seen to play a key rôle in rigorous statistical mechanics.

3. FURTHER DIRECTIONS

Detailed in the Research Project and Research Vision chapters.

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